

# Strongly Almost Periodic Sequences under Finite Automata Mappings

Yuri Pritykin

February 1, 2008

## Abstract

The notion of almost periodicity nontrivially generalizes the notion of periodicity. Strongly almost periodic sequences (=uniformly recurrent infinite words) first appeared in the field of symbolic dynamics, but then turned out to be interesting in connection with computer science. The paper studies the class of eventually strongly almost periodic sequences (i. e., becoming strongly almost periodic after deleting some prefix). We prove that the property of eventual strong almost periodicity is preserved under the mappings done by finite automata and finite transducers. The class of almost periodic sequences includes the class of eventually strongly almost periodic sequences. We prove this inclusion to be strict.

## 1 Introduction

Strongly almost periodic sequences (=uniformly recurrent infinite words) were studied in the works of Morse and Hedlund [1, 2] and of many others (for example see [4]). This notion first appeared in the field of symbolic dynamics, but then turned out to be interesting in connection with computer science.

Evidently, the class of finite automata mappings of strongly almost periodic sequences (for definitions see below) contains the class of eventually strongly almost periodic sequences, i. e., becoming strongly almost periodic after deleting some prefix. Indeed, we use finite automaton with delay to get the sequence  $a\omega$  from the strongly almost periodic sequence  $\omega$ : this automaton keeps the string  $a$  in memory, first outputs this string and then outputs the input sequence with delay  $|a|$  (always remembering last  $|a|$  symbols of the sequence). The main result of the article (Theorem 2) states the equality of the classes. In other words, Theorem 2 says that finite automata preserve

the property of eventual strong almost periodicity. In the last section we consider the generalized version of finite automaton — finite transducer — and prove the same statement for it.

The notion of almost periodic sequence was studied in [5]. The authors prove that the class of almost periodic sequences is also closed under finite automata mappings. It can easily be checked that the class of almost periodic sequences contains the class of eventually strongly almost periodic sequences. We prove this inclusion to be strict (Theorem 1).

Let  $A$  be a finite alphabet with at least two symbols. Consider the sequences over this alphabet — mappings  $\omega: \mathbb{N} \rightarrow A$  (where  $\mathbb{N} = \{0, 1, 2, \dots\}$ ). Denote by  $A^*$  the set of all finite strings over  $A$  including the empty string  $\Lambda$ . If  $i \leq j$  are natural, denote by  $[i, j]$  the segment of natural numbers with ends in  $i$  and  $j$ , i. e., the set  $\{i, i+1, i+2, \dots, j\}$ . Also denote by  $\omega[i, j]$  the segment of the sequence  $\omega$  — the string  $\omega(i)\omega(i+1) \dots \omega(j)$ . A segment  $[i, j]$  is an occurrence of a string  $x \in A^*$  in a sequence  $\omega$  if  $\omega[i, j] = x$ . Denote by  $|x|$  the length of the string  $x$ . We imagine the sequences going horizontally from the left to the right, so we use terms “to the right” or “to the left” to talk about greater and smaller indices respectively.

## 2 Almost periodicity

A sequence  $\omega$  is called *almost periodic* if for any string  $x$  occurring in the sequence infinitely many times there exists a number  $l$  such that any segment of  $\omega$  of length  $l$  contains at least one occurrence of  $x$ . We denote the class of these sequences by  $\mathcal{AP}$ .

A sequence  $\omega$  is called *strongly almost periodic* if for any string  $x$  occurring in the sequence at least once there exists a number  $l$  such that any segment of  $\omega$  of length  $l$  contains at least one occurrence of  $x$  (and therefore  $x$  occurs in  $\omega$  infinitely many times). Denote by  $\mathcal{SAP}$  the class of these sequences.

For convenience we introduce one additional definition: a sequence  $\omega$  is *eventually strongly almost periodic* if  $\omega = a\nu$  for some  $\nu \in \mathcal{SAP}$  and  $a \in A^*$ . The class of these sequences we denote by  $\mathcal{EAP}$ .

Every eventually strongly almost periodic sequence is obviously almost periodic. Let us show that  $\mathcal{EAP}$  is a proper subclass of  $\mathcal{AP}$ .

**Theorem 1.** *There exists a binary sequence  $\omega$  such that  $\omega \in \mathcal{AP}$ , but  $\omega \notin \mathcal{EAP}$ .*

*Proof.* Construct a sequence of binary strings  $a_0 = 1$ ,  $a_1 = 10011$ ,  $a_2 = 1001101100011001001110011$ , and so on, by this rule:

$$a_{n+1} = a_n \bar{a}_n \bar{a}_n a_n a_n,$$

where  $\bar{x}$  is a string obtained from  $x$  by changing every 0 to 1 and vice versa. Put

$$c_n = \underbrace{a_n a_n \dots a_n}_{10}$$

and

$$\omega = c_0 c_1 c_2 c_3 \dots$$

Prove that  $\omega$  is a required one.

The length of  $a_n$  is  $5^n$ , so the length of  $c_0 c_1 \dots c_{n-1}$  is  $10(1 + 5 + \dots + 5^{n-1}) = \frac{5}{2}(5^n - 1)$ . By definition, put

$$l_n = \frac{5}{2}(5^n - 1) = |c_0 c_1 \dots c_{n-1}|.$$

Show that  $\omega$  is almost periodic. Suppose  $x \neq \Lambda$  occurs in  $\omega$  infinitely many times. Take  $n$  such that  $|x| < 5^n$ . Suppose  $[i, j]$  is an occurrence of  $x$  in  $\omega$  such that  $i \geq l_n$ . By construction, for any  $k$  we can consider the part of  $\omega$  starting from the position  $l_k$  as a concatenation of strings  $a_k$  and  $\bar{a}_k$ . Thus by definition of  $i$  the string  $x$  is a substring of one of four strings  $a_n a_n$ ,  $a_n \bar{a}_n$ ,  $\bar{a}_n a_n$ ,  $\bar{a}_n \bar{a}_n$ . Note that the string 10011 contains all strings of length two 00, 01, 10, 11, so the string  $a_{n+1}$  contains each of  $a_n a_n$ ,  $a_n \bar{a}_n$ ,  $\bar{a}_n a_n$ ,  $\bar{a}_n \bar{a}_n$ . So,  $x$  is a substring of  $a_{n+1}$ . Similarly,  $x$  is a substring of  $\bar{a}_{n+1}$ . If the segment of length  $2|a_{n+1}|$  occurs in the sequence to the right of  $l_{n+1}$ , then  $a_{n+1}$  or  $\bar{a}_{n+1}$  occurs in this segment. Hence for  $l = \frac{5}{2}(5^{n+1} - 1) + 2 \cdot 5^{n+1}$  it is true, that on every segment of length  $l$  there exists an occurrence of  $x$ .

Prove now that for any  $n > 0$  the string  $c_n$  does not occur in  $\omega$  to the right of  $l_{n+1}$ . In this case,  $c_n$  occurs finitely many times in the sequence obtained from  $\omega$  by deleting some prefix of the length at most  $l_n$ , i. e., this sequence is not strongly almost periodic. Therefore  $\omega$  is not eventually strongly almost periodic.

Let  $\nu$  be the sequence obtained from  $\omega$  by deleting the prefix of the length  $l_{n+1}$ . As above, for each  $k$ ,  $1 \leq k \leq n+1$ ,  $\nu$  is a concatenation of strings  $a_k$  and  $\bar{a}_k$ . Assume  $c_n$  occurs in  $\nu$  and let  $[i, j]$  be one of this occurrences. For  $n > 0$  the string  $c_n$  begins with  $a_1$ , hence  $[i, i+4]$  is an occurrence of  $a_1$  in  $\nu$ . We see that  $a_1 = 10011$  occurs in  $a_1 a_1 = 1001110011$ ,  $a_1 \bar{a}_1 = 1001101100$ ,  $\bar{a}_1 a_1 = 0110010011$  or  $\bar{a}_1 \bar{a}_1 = 0110001100$  only in zeroth or fifth position.

Thus  $5|i$ , i. e.,  $\nu$  and  $c_n$  can be considered as constructed of “letters”  $a_1$  and  $\bar{a}_1$ , and we assume that  $c_n$  occurs in  $\nu$ . Now it is easy to prove by induction on  $m$  that  $5^m|i$  for  $1 \leq m \leq n$ , i. e., we can consider  $\nu$  and  $c_n$  as constructed of “letters”  $a_m$  and  $\bar{a}_m$ , and assume that  $c_n$  occurs in  $\nu$  (the base for  $m = 1$  is already proved, but we can repeat the same argument changing 1 and 0 to  $a_m$  and  $\bar{a}_m$  and taking into account that  $c_n$  begins with  $a_m$  for each  $1 \leq m \leq n$ ).

Therefore we have shown that  $5^n|i$ , i. e., if we consider  $\nu$  and  $c_n$  to be constructed of “letters”  $a_n$  and  $\bar{a}_n$ , then  $c_n = \underbrace{a_n a_n \dots a_n}_{10}$  occurs in  $\nu$ .

However there exists an occurrence of “five-letter” string  $a_{n+1}$  or  $\bar{a}_{n+1}$  on each segment of 10 consequent “letters”  $a_n$  and  $\bar{a}_n$  in  $\nu$ , and  $\bar{a}_n$  occurs in this string. This is a contradiction.  $\square$

Moreover, it is quite easy to modify the proof and to construct continuum of sequences in  $\mathcal{AP} \setminus \mathcal{EAP}$ . For example, for each sequence  $\tau: \mathbb{N} \rightarrow \{9, 10\}$  we can construct  $\omega_\tau$  in the same way as in the proof of Theorem 1, but instead of  $c_n$  we take

$$c_n^{(\tau)} = \underbrace{a_n a_n \dots a_n}_{\tau(n)}.$$

Obviously, all  $\omega_\tau$  are different for different  $\tau$  and hence there exists continuum of various  $\tau$ .

In conclusion, we can remark that, as it is shown in [5], there exists continuum of different sequences in  $\mathcal{SAP}$  too.

### 3 Finite automata mappings

*Finite automaton* is a tuple  $F = \langle A, B, Q, \tilde{q}, f \rangle$ , where  $A$  and  $B$  are finite sets called input and output alphabets respectively,  $Q$  is a finite set of states,  $\tilde{q} \in Q$  is the initial state, and  $f: Q \times A \rightarrow Q \times B$  is the translation function. We say that the sequence  $\langle p_n, \beta(n) \rangle_{n=0}^\infty$ , where  $p_n \in Q$ ,  $\beta(n) \in B$ , is the automaton mapping of the sequence  $\alpha$  over alphabet  $A$ , if  $p_0 = \tilde{q}$  and for each  $n$  we have  $\langle p_{n+1}, \beta(n) \rangle = f(p_n, \alpha(n))$ . Thus we say that  $F$  outputs the sequence  $\beta$  and denote this output by  $F(\alpha)$ . If  $[i, j]$  is an occurrence of the string  $x$  in the sequence  $\alpha$ , and  $p_i = q$ , then we say that automaton  $F$  comes to this occurrence of  $x$  in the state  $q$ .

In [5] the following statement was proved: *if  $F$  is a finite automaton, and  $\omega \in \mathcal{AP}$ , then  $F(\omega) \in \mathcal{AP}$ .*

This theorem can be expanded.

**Theorem 2.** *If  $F$  is a finite automaton, and  $\omega \in \mathcal{EAP}$ , then  $F(\omega) \in \mathcal{EAP}$ .*

*Proof.* Obviously, it is enough to prove the theorem for  $\omega \in \mathcal{SAP}$ , as every eventually strongly almost periodic sequence keeps this property after addition a prefix.

Thus let  $\omega \in \mathcal{SAP}$ . By the previous statement,  $F(\omega) \in \mathcal{AP}$ . Suppose  $F(\omega)$  is not eventually strongly almost periodic. It means that for any natural  $N$  there exists a string that occurs in  $F(\omega)$  after position  $N$  and does not occur to the right of it. Indeed, if we remove the prefix  $[0, N]$  from  $F(\omega)$ , we do not get strongly almost periodic sequence, hence there exists a string occurring in this sequence only finitely many times. Then take the rightmost occurrence.

Let  $[i_0, j_0]$  be the rightmost occurrence of a string  $y_0$  in  $F(\omega)$ . For some  $l_0$  the string  $x_0 = \omega[i_0, j_0]$  occurs in every segment of the length  $l_0$  in  $\omega$  (by the property of strong almost periodicity). If  $F$  comes to  $i_0$  in the state  $q_0$ , then  $F$  never comes to righter occurrences of  $x$  in the state  $q_0$  because in this case automaton outputs  $y_0$  completely.

Now let  $[r, s]$  be the rightmost occurrence of some string  $a$  in  $F(\omega)$ , where  $r > i_0 + l_0$ . On the segment  $\omega[r - l_0, r]$  there exists an occurrence  $[r', s']$  of the string  $x_0$ . By definition of  $r$  we have  $r' > i_0$ . Thus assume

$$i_1 = r', \quad j_1 = s, \quad x_1 = \omega[i_1, j_1], \quad y_1 = F(\omega)[i_1, j_1].$$

Since  $a$  does not occur in  $F(\omega)$  to the right of  $r$ , then  $y_1$  does not occur in  $F(\omega)$  to the right of  $i_1$ , for it contains  $a$  as a substring. Therefore if the automaton comes to the position  $i_1$  in the state  $q_1$ , then it never comes to righter occurrences of  $x_1$  in the state  $q_1$ . Since  $x_1$  begins with  $\omega[r', s'] = x_0$ , and  $r' > i_0$ , we get  $q_1 \neq q_0$ . We have found the string  $x_1$  such that automaton  $F$  never comes to occurrences of  $x_1$  to the right of  $i_1$  in the state  $q_0$  or  $q_1$ .

Let  $m = |Q|$ . Arguing as above, for  $k < m$  we construct the strings  $x_k = \omega[i_k, j_k]$  and corresponding different states  $q_k$ , such that  $F$  never comes to occurrences of  $x_k$  in  $\omega$  to the right of  $i_k$  in the states  $q_0, q_1, \dots, q_k$ . For  $k = m$  we have a contradiction.  $\square$

## 4 Finite transducers

Let  $A$  and  $B$  be finite alphabets. The mapping  $h: A^* \rightarrow B^*$  is called *homomorphism*, if for any  $u, v \in A^*$  we have  $h(uv) = h(u)h(v)$ . Clearly, any homomorphism is fully determined by its values on one-character strings.

Let  $\omega$  be the sequence over the alphabet  $A$ . By definition, put

$$h(\omega) = h(\omega(0))h(\omega(1))h(\omega(2)) \dots$$

Suppose  $h: A^* \rightarrow B^*$  is a homomorphism,  $\omega$  is an almost periodic sequence over  $A$ . In [5] it was shown that if  $h(\omega)$  is infinite, then it is almost periodic. Thus it is obvious, that if  $\omega$  is strongly almost periodic, and  $h(\omega)$  is infinite, then  $h(\omega)$  is also strongly almost periodic. Indeed, it is enough to show that any  $v$  occurring in  $h(\omega)$  occurs infinitely many times. However there is some string  $u$  occurring in  $\omega$  such that  $h(u)$  contains  $v$ , but by the definition of strong almost periodicity  $u$  occurs in  $\omega$  infinitely many times. Evidently, for  $\omega \in \mathcal{EAP}$  we have  $h(\omega) \in \mathcal{EAP}$ , if  $h(\omega)$  is infinite.

Now we modify the definition of finite automaton, allowing it to output any string (including the empty one) over output alphabet reading only one character from input. This modification is called *finite transducer* (see [3]). Formally, we only change the definition of translation function. Now it has the form

$$f: Q \times A \rightarrow Q \times B^*.$$

If the sequence  $\langle p_n, v_n \rangle_{n=0}^\infty$ , where  $p_n \in Q$ ,  $v_n \in B^*$ , is the mapping of  $\alpha$ , then the output is the sequence  $v_0 v_1 v_2 \dots$ .

Actually, we can decompose the mapping done by finite transducer into two: the first one is a finite automaton mapping and another is a homomorphism. Each of these mappings preserves the class  $\mathcal{AP}$ , so we get the corollary: finite transducers map almost periodic sequences to almost periodic. Similarly, by Theorem 2 and arguments above we also get the following statement: finite transducers map eventually strongly almost periodic sequences to eventually strongly almost periodic (if the output is infinite).

**Acknowledgements.** The author is grateful to Alexei Semenov and Andrej Muchnik for the help in the work and also to the participants of Kolmogorov seminar for useful discussions.

## References

- [1] M. Morse, G. A. Hedlund. *Symbolic dynamics*. American Journal of Mathematics, 60, pp. 815–866, 1938.
- [2] M. Morse, G. A. Hedlund. *Symbolic dynamics II: Sturmian trajectories*. American Journal of Mathematics, 62, pp. 1–42, 1940.

- [3] A. Weber. *On the valuedness of finite transducers*. Acta Informatica, 27, pp. 749–780, 1989.
- [4] J. Cassaigne. *Recurrence in infinite words*. Proceedings of the 18th Symposium on Theoretical Aspects of Computer Science (STACS 2001), Springer Verlag, 1–11, 2001.
- [5] An. Muchnik, A. Semenov, M. Ushakov. *Almost periodic sequences*. Theoretical Computer Science, vol. 304, pp. 1–33, 2003.